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Note

Proof of a conjecture of Dirac concerning 4-critical planar graphs

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Abstract

The conjecture of Dirac that every 4-critical planar graph has a vertex of degree at most 4 was recently proved by Koester. In this article a different proof is given.

A graph G is said to be k-critical if it has chromatic number k but every proper subgraph of G is $(k-1)$-colorable.

Approximately 30 years ago, Dirac [1] conjectured that every 4-critical planar graph has a vertex of degree at most 4.

Recently, the conjecture of Dirac was settled by Koester [2]. A key ingredient of Koester's proof is a theorem of Stiebitz [4] concerning the maximal number of triangles a 4-critical planar graph may contain and whose proof involves algebraic methods. The object of this paper is to give another proof of Dirac's conjecture along different lines.

Theorem 1. *Every 4-critical planar graph has a vertex of degree at most 4.*

Proof. Suppose the theorem is false and let $G=(V,E)$ be a 4-critical planar graph in which each vertex has degree at least 5. Fix a plane drawing of G. Let n, m and f denote the number of vertices, edges and faces of G and let f_i denote the number of i-faces, that is, the number of faces with i vertices. Let $d_G(v)$ denote the degree of the vertex v.

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We have the following well known relations:

(1) $n-m+f=2$ (Euler's formula),

(2) $\sum d_G(v)=\sum i f_i=2m$.

We associate with G a bipartite graph $\hat{G}$ whose parts are V and the set $\mathscr{F}$ of nontriangular faces of G. The edges of $\hat{G}$ are defined as follows: join $v\in V$ to $F\in\mathscr{F}$ if v is incident with F in G. We now develop some properties of G and $\hat{G}$.

Property 1. *The number of edges of* $\hat{G}$ is $\sum_{i\geqslant 4} i f_i$.

Proof. This is clear from $d_{\hat{G}}(F)=|F|$ for all $F\in\mathscr{F}$. □

Property 2. *For each* $v\in V$, $d_{\hat{G}}(v)\geqslant 1$.

Proof. Suppose that some $v\in V$ is incident with only triangular faces and let W denote the wheel with hub v. Since W is a proper subgraph of G, $d_G(v)$ is even. Delete an edge ab of the rim of W. Then $G-ab$ is 3-colorable and in any 3-coloring the veritces of the rim must be colored alternately in two colors and a and b must be assigned different colors. However, this implies that G is 3-colorable. Thus, each $v\in V$ is incident with at least one nontriangular face and Property 2 follows. □

Property 3. *If in* G, *a vertex* v *is incident with exactly one* 4-*face and if all other faces incident with* v *are triangular, then* $d_G(v)$ *is even.*

Proof. Suppose that this is not the case and let the neighbors of v be $v_0, v_1, v_2, \ldots, v_{2k}$. Let v, v_0, v_{2k} and u be the vertices of the 4-face. Since G is 4-critical, $G-uv_0$ is 3-colorable. In any 3-coloring, $v_0, v_2, v_4, \ldots, v_{2k}$ are assigned the same color, say red. Since uv_{2k} is an edge of $G-uv_0$, u cannot be colored red. But this yields a 3-coloring of G. This establishes Property 3. □

Partition V into three sets V_1, V_2 and V_3 as follows:

$$V_1=\{v\colon v\in V, d_G(v)=5 \text{ and } d_{\hat{G}}(v)=1\},$$

$$V_2=\{v\colon v\in V, d_G(v)=5 \text{ and } d_{\hat{G}}(v)\geqslant 2\},$$

$$V_3=\{v\colon v\in V, d_G(v)\geqslant 6\}.$$

Property 4. $|V_3|\leqslant 2m-5n$.

Proof. We have, by (2),

$$2m=\sum_{v\in V} d_G(v)=\sum_{v\in V_3} d_G(v)+\sum_{v\notin V_3} d_G(v)\geqslant 6|V_3|+5(n-|V_3|)=5n+|V_3|$$

and Property 4 follows. □

For $i \geqslant 5$, let

$$V_i = \{v\colon v \in V_1, v \text{ is incident with exactly one } i\text{-face}\}.$$

Note that the definition makes sense when $i=4$. However, V_4 is empty, by Property 3. Also, the sets $V_5, V_6, \ldots$ are pairwise disjoint and their union is V_1.

Property 5. $\sum_{i\geqslant 4} if_i \geqslant 2n - |V_3| - \sum_{i\geqslant 5} |V_i|$.

Proof. The number of edges of $\hat{G}$ is at least

$$\begin{aligned} |V_1| + 2|V_2| + |V_3| &= |V_1| + 2(n - |V_1| - |V_3|) + |V_3| \\ &= 2n - |V_3| - |V_1| = 2n - |V_3| - \sum_{i\geqslant 5} |V_i|, \end{aligned}$$

and Property 5 now follows from Property 1. □

Property 6. $3f_5 \geqslant |V_5|$.

Proof. Suppose that there is a 5-face with consecutive vertices u, v, w, x, y such that $u, v, w, x \in V_5$ and $y \notin V_5$. Then G must contain, as a proper subgraph, the graph H in Fig. 1. However, H is easily seen to be 4-chromatic. Thus no four vertices of V_5 are incident with a 5-face. This implies Property 6. □

Property 7. *For* $i \geqslant 6$, $(i-1)f_i \geqslant |V_i|$.

Proof. Suppose there exist vertices $v_1, v_2, \ldots, v_i$ that are consecutive vertices of an i-face. Then G must contain, as a proper subgraph, the graph shown in Fig. 2. $G - wu_1$ has a 3-coloring in which w and u_1 are assigned the same color, say red. In this 3-coloring $u_2, u_3, \ldots, u_i$ must also be colored red, but this is not the case since u_i and w are adjacent. Thus no i vertices of V_i are vertices of an i-face and Property 7 follows. □

Now we can complete the proof of Theorem 1. We have

$$\begin{aligned} 2m &= \sum_{i\geqslant 3} if_i, \quad \text{by (2)}, \\ &= 3f + \tfrac{1}{4}\left\{ \sum_{i\geqslant 4} if_i + \sum_{i\geqslant 5} (3i-12) f_i \right\} \\ &\geqslant 3(m-n+2) + \tfrac{1}{4}\left\{ 2n - |V_3| - \sum_{i\geqslant 5} |V_i| + \sum_{i\geqslant 5} (3i-12) f_i \right\}, \\ &\quad \text{by (1) and Property 5}, \\ &= 3m - \tfrac{5}{2}n - \tfrac{1}{4}|V_3| + 6 + \tfrac{1}{4}\left\{ 3f_5 - |V_5| + \sum_{i\geqslant 6} ((3i-12)f_i - |V_i|) \right\}. \end{aligned}$$

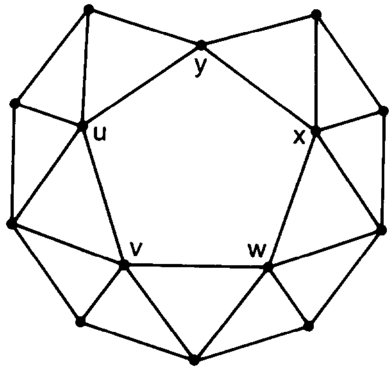

Fig. 1.

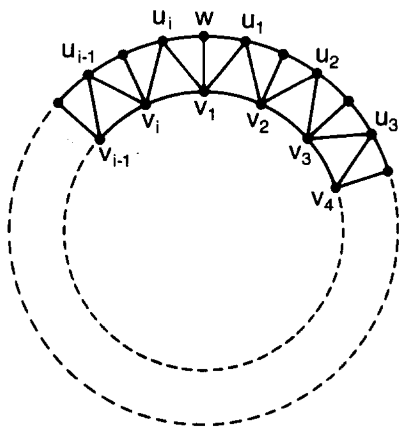

Fig. 2.

By Property 6, $3f_5-|V_5|\geqslant 0$ and by Property 7, $(3i-12)f_i-|V_i|\geqslant((2i-11)/(i-1))|V_i|\geqslant 0$ for $i\geqslant 6$. We then get

$$2m-5n\leqslant\tfrac{1}{2}|V_3|-12.$$

However, this is easily seen to violate Property 4. The theorem follows. □

We mention that Dirac made the stronger conjecture that every 4-critical planar graph has a vertex of degree 3, but this was shown to be false by Koester [3].

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